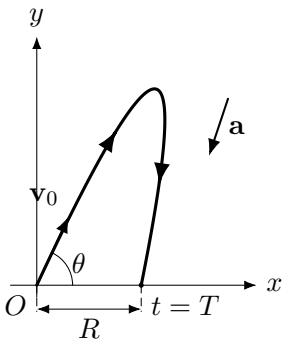
1. A particle is launched from the origin with initial speed v_0, at an angle $0 < \theta < \pi/2$ above the positive x-axis. Throughout its motion, its acceleration is constant and given as:

$$\mathbf{a} = -\frac{g}{3}\hat{\mathbf{i}} - g\hat{\mathbf{j}},$$

where $g > 0$. A possible trajectory is shown below.



(a) Write the particle's position vector $\mathbf{r}(t)$ in unit-vector notation. [5 Pts.]

Initial velocity components: $v_{0x} = v_0\cos\theta, \qquad v_{0y} = v_0\sin\theta$

$a_x = -g/3$ and $a_y = -g$

Using, $x(t) = x_0 + v_{0x}t + \frac{1}{2}a_xt^2$, and $y(t) = y_0 + v_{0y}t + \frac{1}{2}a_yt^2$

$$\Rightarrow \mathbf{r}(t) = \left(v_0\cos\theta\, t - \frac{g}{6}t^2\right)\hat{\mathbf{i}} + \left(v_0\sin\theta\, t - \frac{g}{2}t^2\right)\hat{\mathbf{j}}$$

(b) Find the time $T > 0$ at which the particle crosses the x-axis again. [5 Pts.]

Set $y(T) = 0$:

$$v_0\sin\theta\cdot T - \frac{g}{2}T^2 = 0 \implies T\left(v_0\sin\theta - \frac{g}{2}T\right) = 0.$$

Discarding $T = 0$:

$$T = \frac{2v_0\sin\theta}{g}$$

(c) Find the range R, defined as the horizontal displacement $x(T)$ at the moment of return to the x-axis. [5 Pts.] Range $R = x(T)$

$$R = v_0\cos\theta\cdot T - \frac{g}{6}T^2 = v_0\cos\theta\cdot\frac{2v_0\sin\theta}{g} - \frac{g}{6}\cdot\frac{4v_0^2\sin^2\theta}{g^2} = \frac{2v_0^2\sin\theta\cos\theta}{g} - \frac{2v_0^2\sin^2\theta}{3g}$$

$$R = \frac{v_0^2}{g}\left(\sin 2\theta - \frac{1-\cos 2\theta}{3}\right)$$

(d) Find the launch angle that maximizes R. [10 Pts.]

Differentiate with respect to θ and set to zero:

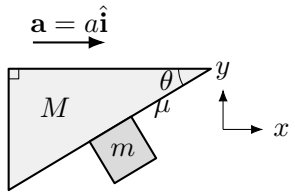
$$\frac{dR}{d\theta} = \frac{v_0^2}{g}\left(2\cos 2\theta - \frac{2\sin 2\theta}{3}\right) = 0$$

$$\cos 2\theta = \frac{\sin 2\theta}{3} \implies \tan 2\theta = 3.$$

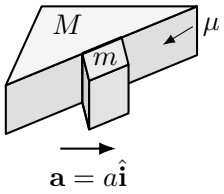
$$2\theta = \arctan 3 \implies \theta_{\max} = \tfrac{1}{2}\arctan 3$$

2. A triangular prism (mass M) and a cube (mass m) rest on a frictionless table, as shown in figure. The cube's face contacts the hypotenuse face of the prism; the coefficient of friction between them is μ (static and kinetic). The hypotenuse makes angle θ with the x-axis. A horizontal force $\mathbf{F}$ gives the prism a constant acceleration $\mathbf{a} = a\hat{\mathbf{i}}$ directed toward the cube as shown in the Figure. Neither block rotates or tips. Consider only while the blocks remain in contact.

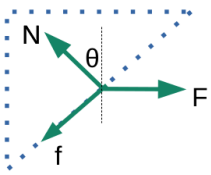
Top view

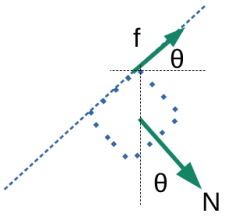


Perspective view



(a) Draw free-body diagrams for the prism and the cube, showing only the horizontal forces. [5 Pts.]





(b) Determine the condition on μ and θ for the cube to accelerate without slipping relative to the prism. [5 Pts.]

If the cube does not slip, both blocks accelerate at $a\hat{\mathbf{i}}$. Apply Newton's second law to the cube:

$$x: \quad N\sin\theta + f\cos\theta = ma \tag{1}$$

$$y: \quad -N\cos\theta + f\sin\theta = 0 \quad \Rightarrow f = N\cot\theta \tag{2}$$

Friction is positive, meaning it acts up the face. This makes sense: the cube tends to slide *down* the face, so static friction resists this. **No-slip condition** $f \le \mu N$:

$$\mu \ge \cot\theta = \frac{\cos\theta}{\sin\theta}$$

(c) Find the x- and y-components of the external force $\mathbf{F}$ required to produce the prescribed acceleration of the prism when the cube does not slip relative to it. [5 Pts.]

Newton's second law for the whole system (only external horizontal force is $\mathbf{F}$; table is frictionless) yields

$$F_x = (M+m)a$$

The internal forces between the blocks have zero net y-component on the system when

there is no slip $\Rightarrow F_y = 0$ because

$$\vec{F} = (M+m)a\hat{\mathbf{i}}$$

(d) If the condition in part (b) is not satisfied, find the acceleration of the cube **relative to the prism**. Give its magnitude and direction. [10 Pts.]

When $\mu < \cot\theta$, friction is insufficient to maintain no-slip. The cube slides down the face and stays in contact with the face. Apply Newton's 2nd law to the cube (friction in $+\hat{t}$):

$$x: \quad N\sin\theta + f cos\theta = ma_x^m \tag{6}$$

$$y: \quad -N\cos\theta + f\sin\theta = ma_y^m \tag{7}$$

$$\Rightarrow N = m(\sin\theta a_x - \cos\theta a_y)$$

$$f = m(\cos\theta a_x + \sin\theta a_y)$$

Kinetic friction $f = \mu N$. The prism accelerates at $(a, 0)$. Relative acceleration of cube with respect to prism: $a_{xr} = a_x - a$ and $a_{yr} = a_y - 0$. Using the geometry: $\frac{|a_y|}{|a-a_x|} = \tan\theta \Rightarrow a_y = \tan\theta(a_x - a)$

$$N = m(\sin\theta a_x + \cos\theta\tan\theta(a_x - a))$$

$$\mu N = m(\cos\theta a_x - \sin\theta\tan\theta(a_x - a)$$

$$\Rightarrow \mu m(\sin\theta a_x + \sin\theta a - \sin\theta a_x) = m(\cos\theta a_x - \frac{\sin^2\theta}{\cos\theta}a + \frac{\sin^2\theta}{\cos\theta}a_x)$$

Then,

$$a_x = a\cos\theta\,\sin\theta(\mu + \tan\theta) \quad \Rightarrow a_{xr} = a_x - a = a\cos\theta\,(\mu\sin\theta - \cos\theta)$$

$$a_{yr} = a_y - 0 = a\sin\theta(\mu\sin\theta - \cos\theta)$$

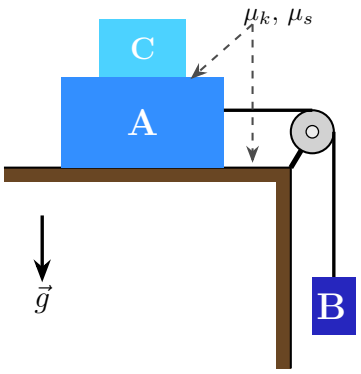
$$|\mathbf{a}^{\text{rel}}| = a\,|\mu\sin\theta - \cos\theta|\sqrt{\cos^2\theta + \sin^2\theta} = a(\cos\theta - \mu\sin\theta)$$

Down the hypotenuse face

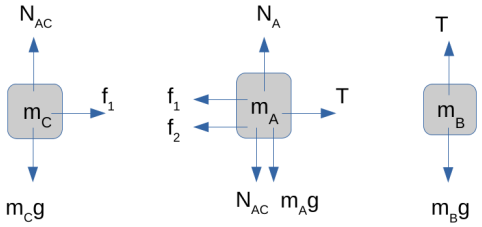
Positive since $\mu < \cot\theta \Rightarrow \cos\theta > \mu\sin\theta$

- At $\mu = \cot\theta$: $a_{\text{rel}} = 0$ — consistent with the no-slip boundary in part (b). ✓
- At $\mu = 0$ (frictionless): $a_{\text{rel}} = a\cos\theta$ — maximum sliding, as expected. ✓

3. Blocks A ($2M$), B (M), and C (m) are arranged as shown. A and B are connected by a massless cord over a frictionless pulley. C rests on top of A. Both interfaces (table–A and A–C) have the same coefficients of friction μ_k and μ_s and both $< 1/2$.



(a) Draw free-body diagrams for each block. [6 Pts.]



(b) Determine the minimum mass m of block C to keep A from sliding. [7 Pts.]

When system is stationary, B tends to pull A in $+x$. Tension: $T = Mg$ (B in equilibrium).
Friction: $f = f_1 + f_2 = T = Mg$
Normal forces: $N_{AC} = mg \quad N_A = N_{AC} + 2Mg = (2M + m)g$.
For A not to slide, $f \leq \mu_s N_A$:

$$\mu_s(2M + m)g \geq Mg$$

$$\mu_s\, m \geq M(1 - 2\mu_s)$$

$$m \geq \frac{M(1 - 2\mu_s)}{\mu_s}$$

Note: Since $\mu_s < 1/2$ the right-hand side is positive, so a finite minimum mass is required.

(c) Block C is suddenly lifted off A. What is the acceleration of block A? [12 Pts.]

After C is removed, B pulls A (via tension; kinetic friction from table opposes A's motion.
Normal force on A: $N = 2Mg$. Kinetic friction on A: $f_k = 2\mu_k Mg$ (in $-x$).
Let a be the common magnitude of acceleration (B downward, A in $+x$):

$$Mg - T = Ma \tag{1}$$

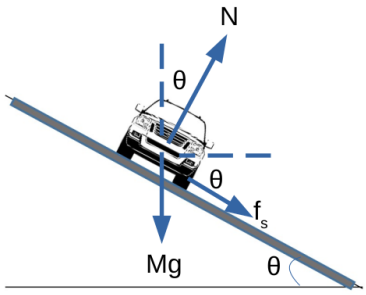
$$T - 2\mu_k Mg = 2Ma \tag{2}$$

Add (1) and (2):

$$Mg - 2\mu_k Mg = 3Ma \qquad \Rightarrow a = \frac{g(1 - 2\mu_k)}{3}$$

Note: The constraint $\mu_k < 1/2$ (given in the problem) ensures $a > 0$, confirming A does accelerate. ✓

4. A highway curve with radius R is banked in such a way that a car traveling at speed v_s can safely turn even on ice (i.e., when there is no friction). The coefficient of static friction for a car is μ_s.



(a) Draw a free-body diagram for a car traveling faster than v_s on the banked curve. [5 Pts.]

(b) Find the banking angle θ such that a car traveling at speed v_s needs no friction to navigate the curve. With no friction, only N and mg act. [5 Pts.]

Vertical equilibrium:

$$N\cos\theta = mg \tag{1}$$

Centripetal (radial) direction:

$$N\sin\theta = \frac{mv_s^2}{R} \tag{2}$$

Divide (2) by (1):

$$\tan\theta = \frac{v_s^2}{Rg} \quad \Rightarrow \quad \theta = \arctan\left(\frac{v_s^2}{Rg}\right)$$

(c) With what maximum speed can the car enter the curve and not skid outward? Express your answer in terms of μ_s, v_s, R, and g. [15 Pts.]

At maximum speed, friction is at its limit $f_s = \mu_s N$, directed down the bank (toward center).

Vertical:

$$N\cos\theta - \mu_s N\sin\theta = mg \implies N = \frac{mg}{\cos\theta - \mu_s\sin\theta} \tag{3}$$

Centripetal:

$$N\sin\theta + \mu_s N\cos\theta = ma_r = \frac{mv^2}{R} \implies N = \frac{mv^2}{R(\sin\theta + \mu_s\cos\theta)} \tag{4}$$

$$\Rightarrow \frac{v^2}{Rg} = \frac{\sin\theta + \mu_s\cos\theta}{\cos\theta - \mu_s\sin\theta} = \frac{\tan\theta + \mu_s}{1 - \mu_s\tan\theta} = \frac{\dfrac{v_s^2}{Rg} + \mu_s}{1 - \mu_s\dfrac{v_s^2}{Rg}} = \frac{v_s^2 + \mu_s Rg}{Rg - \mu_s v_s^2}$$

$$v_{\max} = \sqrt{\frac{Rg(v_s^2 + \mu_s Rg)}{Rg - \mu_s v_s^2}} = v_s\sqrt{\frac{1 + \mu_s Rg/v_s^2}{1 - \mu_s v_s^2/(Rg)}}$$

$\mu_s = 0$: $v_{\max} = v_s$ ✓ (no friction, design speed is the only safe speed)